MATH 3371 **Sample Final Exam**

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Name: KEY

Instructions:

- All answers must be written clearly.
- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).
- You must include all work to receive full credit.

1. Consider a standard deck of 52 cards. What is the probability of a four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

$$\underbrace{\binom{13}{1}}_{\text{4 of a kind Rank}} \cdot \underbrace{\binom{4}{4}}_{\text{Choose all 4}} \cdot \underbrace{\binom{12}{1}}_{\text{Choose Rank}} \cdot \underbrace{\binom{4}{1}}_{\text{choose suit}} \Big/ \binom{52}{5}$$

$$= \frac{13 \cdot 12 \cdot 4}{\binom{52}{5}}$$

2. Consider a roullete wheel consisting of 50 numbers 1 through 50, 0, and 00. If Phan always bets that the outcome will be one of the numbers 1 through 20, what is the probability that

   (a) Phan will lose his first 7 bets,

#s 1-50, 0, 00 = 52 total

$$\mathbb{P}(\text{lose } 1^{st} \text{ 7 bets}) = \frac{32}{52} \cdot \frac{32}{52} \cdot \frac{32}{52} \cdot \frac{32}{52} \cdot \frac{32}{52} \cdot \frac{32}{52} \cdot \frac{32}{52}$$

$$\simeq .033$$

   (b) his first win will occur on his ninth bet?

$$\underbrace{\left(\frac{32}{52}\right)^8}_{\text{lose } 1^{st} \text{ 8th}} \cdot \underbrace{\frac{20}{52}}_{\text{wins on 9th bet}} = \left(\frac{32}{52}\right)^8 \left(\frac{20}{52}\right)$$

3. A manufacturing company sources widgets from three different suppliers (A, B, and C). Based on the company's quality control data, it appears that 3 percent of widgets coming from A are faulty, as are 5 percent of the widgets coming from B, and 2 percent coming from C. Based on recent pnrchasing records, suppliers A, B, and C supply 30 percent, 20 percent, and 50 percent of the company's stock of widgets, respectively.

(a) What is the probability that a random widget from the company's stock is faulty?

$\mathbb{P}(A) = .3, \quad \mathbb{P}(B) = .2, \quad \mathbb{P}(C) = .5$

$F = \{\text{Faulty}\}$, By Law of Total Probability

$\mathbb{P}(F) = \mathbb{P}(F \mid A)\mathbb{P}(A) + \mathbb{P}(F \mid B)\mathbb{P}(B) + \mathbb{P}(F \mid C)\mathbb{P}(C)$

$= (.03)(.3) + (.05)(.2) + (.02)(.5)$

$= \boxed{.029}$

(b) Given that a widget is faulty, what is the probability that it came from supplier C?

$$\mathbb{P}(C \mid F) = \frac{\mathbb{P}(C \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F \mid C)\,\mathbb{P}(C)}{\mathbb{P}(F)}$$

$$= \frac{(.02)(.5)}{.029} = \boxed{.345}$$

(c) Using the definition of independence of events, determine whether the events $F = \{\text{widget is faulty}\}$ and $C = \{\text{widget came from supplier C}\}$ are independent or not.

Not independent, since

$\mathbb{P}(C \mid F) = .345$

$\neq$

$\mathbb{P}(C) = .5$

and if independent then $\mathbb{P}(C \mid F) = \mathbb{P}(C)$.

4. UNH students have designed the new u-phone. They have determined that the lifetime of a U-Phone is given by the random variable X (measured in honrs), with probability density function

$$f(x) = \begin{cases} \frac{10}{x^2} & x \geq 10 \\ 0 & x \leq 10 \end{cases}.$$

(a) Use the PDF to find the probability that the u-phone will last more than 20 hours?

X = life of u-phone

$$P(X \geq 20) = \int_{20}^{\infty} f(x)\,dx = \int_{20}^{\infty} \frac{10}{x^2}\,dx = \int_{20}^{\infty} 10x^{-2}\,dx = 10 \left.\frac{x^{-1}}{-1}\right|_{x=20}^{x=\infty}$$

$$= -10\left[\lim_{x\to\infty} \frac{1}{x} - \frac{1}{20}\right] = -10\left[0 - \frac{1}{20}\right] = \frac{10}{20} = \boxed{\frac{1}{2}}$$

(b) Use the PDF to find the probability that the u-phone will last less than 50 hours.

$X \in [10, \infty)$

$$\therefore P(X \leq 50) = \int_{-\infty}^{50} f(x)\,dx = \int_{10}^{50} \frac{10}{x^2}\,dx$$

because that's the lowest X can ever be.

$$= 10\left[-\frac{1}{x}\right]_{10}^{50} = -10\left[\frac{1}{50} - \frac{1}{10}\right] = -\frac{10}{50} + \frac{10}{10} = -\frac{1}{5} + 1$$

$$= \frac{-1+5}{5} = \boxed{\frac{4}{5}}$$

(c) What is the mean life of the u-phone?

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{10}^{\infty} x \cdot \frac{10}{x^2}\,dx$$

$$= 10\int_{10}^{\infty} \frac{1}{x}\,dx = 10 \ln x \Big|_{x=10}^{x=\infty} = 10\left[\lim_{x\to\infty} \ln x - \ln 10\right]$$

$$= 10\left[\infty - \ln 10\right] = \infty.$$

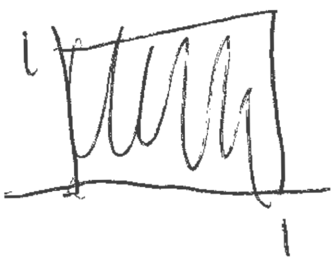
5. You should also know how to answer questions regarding the following distribntions. See study guide, past exams, and past sample exams.
   (a) Binomial
   (b) Poisson
   (c) Exponential
   (d) Normal

6. Suppose the joint density function of the random variables X and Y is

$$f(x,y) = \begin{cases} c\,(x+y) & 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the value of c.

Domain of f



$$1 = \iint_D f\,dy\,dx = \int_0^1 \int_0^1 c\,(x+y)\,dy\,dx$$

$$= c\int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=1} dx$$

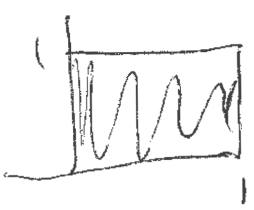
$$= c\int_0^1 \left(x + \frac{1}{2}\right) dx = c\left[\frac{x^2}{2} + \frac{x}{2}\right]_0^1$$

$$= c\left[\frac{1}{2} + \frac{1}{2}\right] = c$$

$$\Rightarrow \boxed{c=1}$$

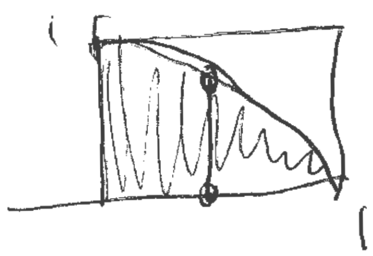
(b) Set up the double integral for $\mathbb{P}\left(X^2+Y^2 \leq 1\right)$. No need to evaluate.

Domain of f



Region

$R = \{(x,y) : x^2+y^2 \leq 1\}$ = Circle of radius 1



$$\boxed{\begin{array}{l} \text{Top } y = \sqrt{1-x^2} \\ \text{Bot } y = 0 \\ 0 \leq x \leq 1 \end{array}}$$

$$\mathbb{P}(X^2+Y^2 \leq 1) = \int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\,dy\,dx$$

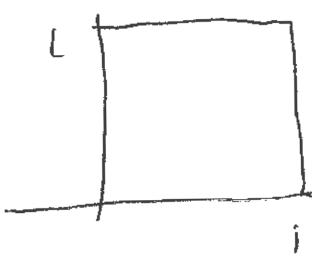
2 ways to do this

$dydx$

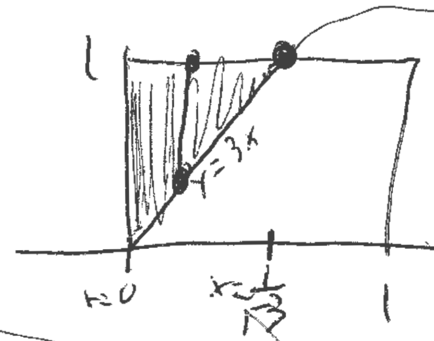
$Y > 3X$

(c) Compute $\mathbb{P}(Y > 3X)$

Domain of f



Region $R = \{(x,y) : y > 3x\}$



y is above line $y = 3x$

$y = 1$ and $y = 3x$ intersect when

$1 = 3x$

$x = \frac{1}{3}$

Top $y = 1$

Bot $y = 3x$

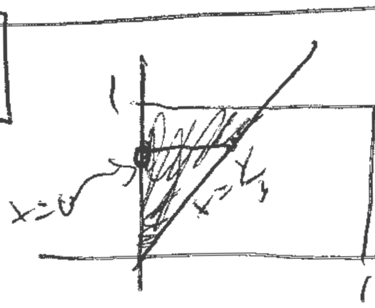
$0 \le x \le \frac{1}{3}$

$$\mathbb{P}(Y > 3X) = \int_0^{1/3}\int_{3x}^{1}(x+y)\,dy\,dx = \int_0^{1/3}\left[xy + \frac{y^2}{2}\right]_{y=3x}^{y=1} dx$$

$$= \int_0^{1/3}\left(x + \frac{1}{2}\right) - \left(3x^2 + \frac{9}{2}x^2\right) dx = \int_0^{1/3}\left(x + \frac{1}{2} - \frac{15}{2}x^2\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x}{2} - \frac{15}{6}x^3\right]_0^{1/3} = \boxed{\frac{(1/3)^2}{2} + \frac{1/3}{2} - \frac{15}{6}(1/3)^3}$$

OR $dxdy$



Right/Left

Right $x = \frac{y}{3}$

Left $x = 0$

$0 \le y \le 1$

$$\boxed{\int_0^1\int_0^{y/3}(x+y)\,dx\,dy}$$

Useful definitions and facts:

Law of Total Probability. If $F_1, \ldots, F_n$ are mutually exclusive events such that they make up the whole sample space, $S = \bigcup_{i=1}^{n} F_i$ then

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \mid F_i)\,\mathbb{P}(F_i).$$

Bayes's Formula. If $F_1, \ldots, F_n$ are mutually exclusive events such that they make up the whole sample space, $S = \bigcup_{i=1}^{n} F_i$ then we have the following conditional probabilities:

$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(E \mid F_j)\,\mathbb{P}(F_j)}{\sum_{i=1}^{n} \mathbb{P}(E \mid F_i)\,\mathbb{P}(F_i)}.$$

for each $j = 1, \ldots, n$.

- **Discrete random variable:**
  - **PMF (Probability Mass Function):** $p_X(x) := \mathbb{P}(X = x)$, (NOTE: some texts may use the notation for $f_X(x) = \mathbb{P}(X = x)$ to denote the PMF)
    - **Properties of a pmf $p(x)$:**
    - Note that we must have $0 < p(x_i) \leq 1$ for $, i = 1, 2, \ldots$. and $p(x) = 0$ for all other values of x can't attain.
    - Also must have

$$\sum_{i=1}^{\infty} p(x_i) = 1.$$

  - **CDF:** $F_X(x) := \mathbb{P}(X \leq x)$.
- **Continuous Random Variables:**
- A random variable X is said to have a **continuous distribution** if there exists a nonnegative function f_X (called the probability distribution function or **PDF**) such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

for every a and b.

  - All **PDFs** must satisfy:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

  - **CDF:** $F_X(x) := \mathbb{P}(X \leq x)$

- **Expected Values:** If $g : \mathbb{R} \to \mathbb{R}$
  - Discrete R.V.: List $X \in \{x_1, x_2, \ldots\}$
    - $\mathbb{E}[g(X)] = \sum_{i=1}^{\infty} g(x_i)\, p_X(x_i)$
  - Continuous R.V.:

* $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

- **Fact:** For continuous R.V we have the following useful relationship
  - Since $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(y)dy$ then by the fundamental theorem of calculus we have

$$F_X'(x) = f_X(x).$$

- How to find the PDF of $Y = g(X)$ where X is the PDF of X.
  - **Step1:** First start by writing the cdf of Y and in terms of F_X:
  - **Step2:** Then use the relation $f_Y(y) = F_Y'(y)$ and take a derivative of the expression obtained in Step 1.

- **Joint Distributions:**
  - **Discrete: joint probability mass(discrete density) function**

$$p(x, y) = \mathbb{P}(X = x, Y = y).$$

    * Some texts may use $f(x, y)$ to denote the PMF.
  - **Continuous:** For random variables X, Y we let $f(x, y)$ be the **joint probability density function,** if

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x, y)dydx.$$

    * Or in general if $D \subset \mathbb{R}^2$ is a region in the plane then

$$\mathbb{P}((X, Y) \in D) = \int\int_D f(x, y)dydx.$$

  - **INDEPENDENCE:**
  - Continuous (discrete) r.v. X, Y are independent if and only if their joint pdf (pmf) can be expressed as

$$f_{X,Y}(x, y) = f_X(x)f_Y(y). \text{ (Continuous Case)},$$
$$p_{X,Y}(x, y) = p_X(x)p_Y(y) \text{ (Discrete Case)}.$$

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